Mathematical Analysis and Generation of Farey Fractals

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Why Fractals?

Fractals have found important applications in the field of Magnetic Resonance Imaging, particularly in the designs of **efficient sampling patterns**. Essentially, we want to capture essential image information with fewer samples.

One such example is the **Farey fractal**, which is generated from a **Farey sequence**. Despite their visual complexity, relatively little is known about their **underlying mathematical properties**. This project aims to explore these properties in greater depth and to develop **computational tools** for generating and analysing Farey fractals.

Ultimately, the goal is to equip researchers with both the **mathematical understanding** and **practical resources** needed to effectively **utilise** these fractals in their own applications.

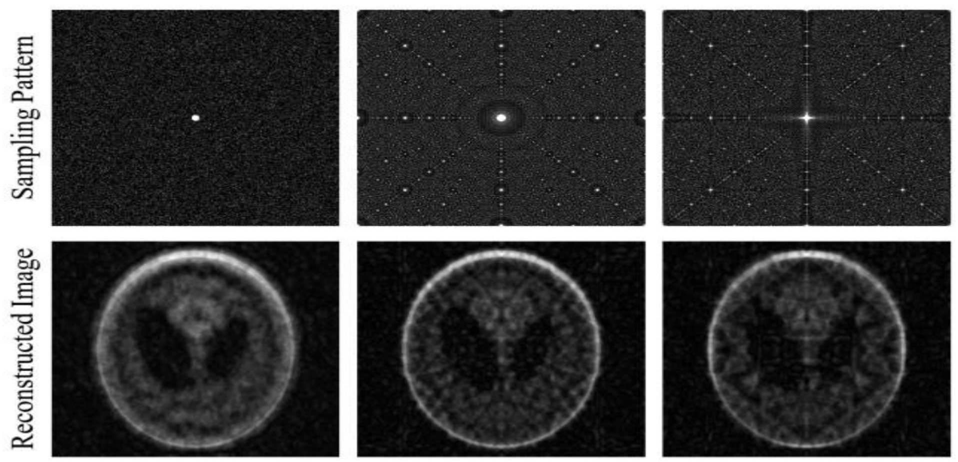

Figure 1: Different sampling pattern associations [1]

What are Fractals?

- **Fractals** are shapes or patterns that display **self-similarity**, meaning their structure repeats at different levels of magnification
- They often arise from **simple rules** that, when repeated, generate **unexpectedly complex** and visually rich behaviour
- Through exploring these patterns, we gain insight into the **mathematical order** hidden within apparent chaos

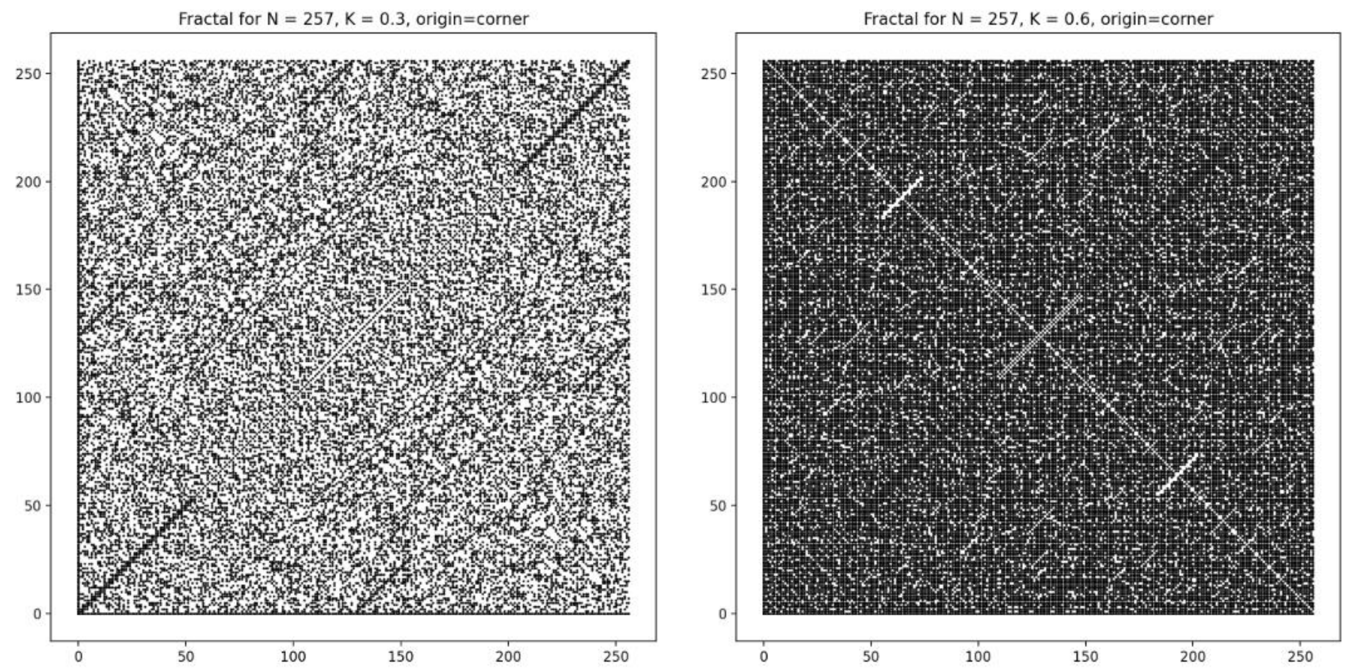

Figure 2: Software-generated Farey fractals, with different K values

Generation Process

1. Create the Farey sequence, F_N.
2. Map each fraction $\frac{a}{b}$ to the point (b, a) on an $N \times N$ grid.
3. Points are mirrored across the horizontal and vertical axis.
4. Sort points by Euclidean distance from the origin (for a consistent growth pattern).
5. Apply the Katz criterion – only retains points satisfying a threshold of $\sqrt{a^2+b^2} \leq K \cdot \max(a, b)$.
6. Each retained point (b, a) generates a full periodic line through $\big((bi) \bmod(N), (ai) \bmod(N)\big),\ 0 \leq i < N$.
7. The resulting points are plotted, to form a Farey fractal.

What are Farey Sequences?

Farey sequences are ordered lists of **reduced fractions** between 0 and 1 with denominators less than or equal to a given integer *n*.

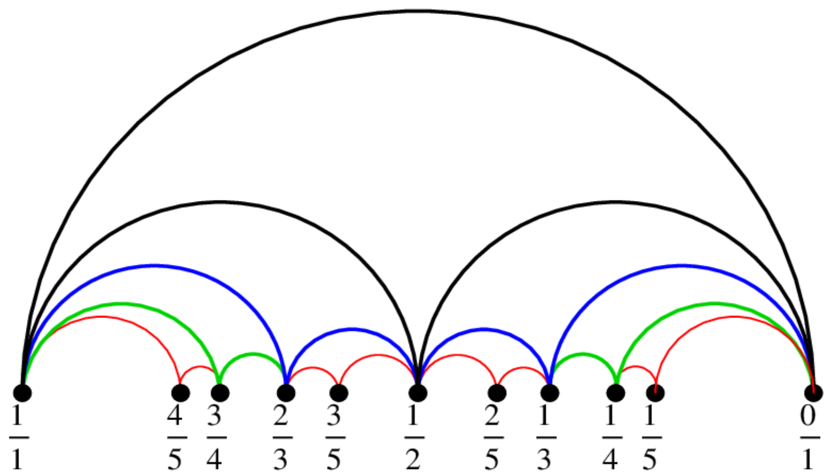

Figure 3: Farey sequence of order 5

Software Development

A **Python-based application** was built using **Streamlit** to generate and visualise **Farey fractals**. The tool enables exploration of how **mathematical parameters**, such as *N* and *K*, affect the resulting structures.

It serves as both a **research platform** for quantitative analysis, and a **visual framework** for investigating fractal geometry and complexity.

Results & Conclusion

- The **mediant insertion property** can be **proven mathematically**, among others, to guide the Farey sequence generation
- **Farey fractals** exhibit clear **self-similarity** and hierarchical structure
- **Fractal dimension** reveals increasing **complexity** with higher *N* and *K*
- Visual patterns suggest **emergent regularities** are present, as well as **distinct geometric symbols**
- Software allows for the **efficient comparison and analysis** of fractal sets

An approximate form of the Hausdorff dimension, in terms of *N* and *K*, was derived to be $D(N,K) \approx 2 - \frac{\log(\frac{1}{K})}{\log(N)}$, and this is currently being checked for accuracy. This work bridges number theory and geometry, offering new insight into how simple rational relationships can generate complex fractal structures.

Acknowledgements

[1] J. M. White, S. Crozier, and S. S. Chandra, "Bespoke Fractal Sampling Patterns for Discrete Fourier Space via the Kaleidoscope Transform," IEEE Signal Processing Letters, vol. 28, pp.2053–2057, 2021, doi: https://doi.org/10.1109/lsp.2021.3116510.

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